

# Ramanujan's Approximation to the $n$ th Partial Sum of the Harmonic Series

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## Abstract

A simple integration by parts and telescopic cancellation leads to a derivation of the first two terms of Ramanujan's asymptotic series for the  $n$ th partial sum of the harmonic series. Kummer's transformation gives three more terms with an explicit error estimate. We also give best-possible estimates of Lodge's approximations.

Entry 9 of Chapter 38 of B. BERNDT's edition of RAMANUJAN's Notebooks, Volume 5 [1, p. 521] reads (in part):

*“Let  $m := \frac{n(n+1)}{2}$ , where  $n$  is a positive integer. Then, as  $n$  approaches infinity,*

$$\sum_{k=1}^n \frac{1}{k} \sim \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - [\dots].”$$

(The entry includes terms up to  $m^{-9}$ .)

Berndt's proof simply verifies (as he himself explicitly notes) that Ramanujan's expansion coincides with the standard EULER expansion

$$\begin{aligned} H_n &:= \sum_{k=1}^n \frac{1}{k} \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - [\dots] \\ &= \ln n + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{n^k} \end{aligned}$$

where  $B_k$  denotes the  $k^{\text{th}}$  BERNOULLI number and  $\gamma := 0.57721 \dots$  is Euler's constant.

However, Berndt does *not* show that Ramanujan's expansion is *asymptotic* in the sense that the error in the value obtained by stopping at any particular stage in Ramanujan's series is less than the next term in the series. Indeed we have been unable to find *any* error analysis of Ramanujan's series.

We therefore offer the following error analysis which shows *the first five terms* of Ramanujan's series to be asymptotic in the sense above. Our methods can be extended to any number of terms in the expansion, but we will limit our presentation to the first five.

Berndt also states that there is no “natural” way to obtain an expansion of  $H_n$  in powers of  $m$ . In fact, our method produces the first two terms of Ramanujan's expansion, namely,  $\frac{1}{12m} - \frac{1}{120m^2}$ , *automatically*, and the later terms by a simple “KUMMER's transformation”: see [4, p. 260].

**Theorem 1.** *Let  $m := \frac{1}{2}n(n+1)$ , where  $n$  is a positive integer. Then there exists a  $\Theta_n$ , with  $0 < \Theta_n < 1$ , for which the following equation is true:*

$$\boxed{\sum_{k=1}^n \frac{1}{k} = \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{\Theta_n}{2310m^5}.}$$

We observe that this expansion of  $H_n$  *does have* the property that the error in the value obtained by stopping at any particular stage in it is less than the next term since the terms alternate in sign and decrease monotonically in absolute value.

*Proof.* We follow a hint from BROMWICH [2, p. 460, Exercise 18]: set

$$\epsilon_n := H_n - \frac{1}{2} \ln[n(n+1)] - \gamma.$$

Then (this is Bromwich's hint),

$$\epsilon_{n-1} - \epsilon_n = \int_0^1 \frac{t^2}{n(n^2 - t^2)} dt.$$

Therefore,

$$\begin{aligned} \epsilon_n &= (\epsilon_n - \epsilon_{n+1}) + (\epsilon_{n+1} - \epsilon_{n+2}) + \cdots \\ &= \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^2}{k(k^2 - t^2)} dt. \end{aligned}$$

Now, integrating by parts and using the partial fraction expansion of  $\frac{1}{k(k^2 - 1)}$ , and then integrating by parts again and using the partial fraction expansion of  $\frac{1}{k(k^2 - 1)^2}$ , we obtain

$$\begin{aligned}
\epsilon_n &= \sum_{k=n+1}^{\infty} \left\{ \frac{1}{3k(k^2-1)} - \frac{2}{3} \int_0^1 \frac{t^4}{k(k^2-t^2)^2} dt \right\} \\
&= \sum_{k=n+1}^{\infty} \left\{ \frac{1}{6} \left( \frac{1}{[k-1]k} - \frac{1}{k[k+1]} \right) - \frac{2}{3} \int_0^1 \frac{t^4}{k(k^2-t^2)^2} dt \right\} \\
&= \frac{1}{6n(n+1)} - \sum_{k=n+1}^{\infty} \left\{ \frac{2}{15k(k^2-1)^2} - \frac{8}{15} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt \right. \\
&\quad \left. - \frac{8}{15} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt \right\} \\
&= \frac{1}{6n(n+1)} - \sum_{k=n+1}^{\infty} \left\{ \frac{2}{15} \left[ \frac{1}{4(k-1)^2} - \frac{1}{2k(k-1)} - \frac{1}{4(k+1)^2} + \frac{1}{2k(k+1)} \right] \right. \\
&\quad \left. - \frac{8}{15} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt \right\} \\
&= \frac{1}{6n(n+1)} - \frac{2}{15} \left( -\frac{1}{2n[n+1]} + \frac{1}{4n^2} + \frac{1}{4[n+1]^2} \right) + \sum_{k=n+1}^{\infty} \frac{8}{15} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt \\
&= \frac{1}{6n(n+1)} - \frac{2}{15} \left( \frac{1}{4n^2[n+1]^2} \right) + \sum_{k=n+1}^{\infty} \frac{8}{15} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt \\
&= \frac{1}{12m} - \frac{1}{120m^2} + \frac{8}{15} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt,
\end{aligned}$$

and we have obtained *the first two terms* of Ramanujan's expansion in powers of  $m$  in a very simple and straightforward manner.

A third integration by parts gives us

$$\frac{8}{15} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt = \frac{8}{105} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^3} - \frac{16}{35} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^8}{k(k^2-t^2)^4} dt.$$

Unfortunately, the series

$$\frac{8}{105} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^3}$$

apparently does not lead to a nice partial fractions telescopic cancellation, and so we need a *new idea*. If we look at the *asymptotic* behavior, as  $n \rightarrow \infty$ , of the error term

$$\frac{8}{15} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^6}{k(k^2-t^2)^3} dt,$$

we observe that the definition of  $m$  implies that  $n^2 \sim 2m$  and therefore the error term is

$$\begin{aligned}
\frac{8}{15} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^6}{k(k^2 - t^2)^3} dt &= \frac{8}{105} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2 - 1)^3} - \frac{16}{35} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^8}{k(k^2 - t^2)^4} dt \\
&\sim \frac{8}{105} \sum_{k=n+1}^{\infty} \frac{1}{k^7} \\
&\sim \frac{8}{105} \cdot \frac{1}{6n^6} \\
&\sim \frac{8}{105} \cdot \frac{1}{48m^3} = \frac{1}{630m^3}
\end{aligned}$$

and the next term in Ramanujan's expansion is indeed  $\frac{1}{630m^3}$ .

The new idea is this. We observe that the asymptotic error term

$$\frac{1}{630m^3} = \frac{8}{105} \cdot \frac{1}{6n^3(n+1)^3}$$

can be represented as *the sum of a telescopic series* as follows:

$$\begin{aligned}
\frac{1}{6n^3(n+1)^3} &= \frac{1}{6n^3(n+1)^3} - \frac{1}{6(n+1)^3(n+2)^3} + \\
&\quad \frac{1}{6(n+1)^3(n+2)^3} - \frac{1}{6(n+2)^3(n+3)^3} + [\dots].
\end{aligned}$$

Therefore, if we *add and subtract* this expansion from the error term series we obtain

$$\begin{aligned}
\frac{8}{105} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2 - 1)^3} &= \frac{8}{105} \frac{1}{6n^3(n+1)^3} + \frac{8}{105} \sum_{k=n+1}^{\infty} \left\{ \frac{1}{k(k^2 - 1)^3} - \left[ \frac{1}{6(k-1)^3 k^3} - \frac{1}{6k^3(k+1)^3} \right] \right\} \\
&= \frac{1}{630m^3} - \frac{8}{315} \sum_{k=n+1}^{\infty} \frac{1}{k^3(k^2 - 1)^3},
\end{aligned}$$

and the error in Ramanujan's expansion takes the form

$$\begin{aligned}
\epsilon_n &:= H_n - \frac{1}{2} \ln(2m) - \gamma \\
&= \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{8}{315} \sum_{k=n+1}^{\infty} \frac{1}{k^3(k^2 - 1)^3} - \frac{16}{35} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^8}{k(k^2 - t^2)^4} dt,
\end{aligned}$$

and we now have *three* terms of Ramanujan's expansion. This technique is a simple example of Kummer's transformation.

To extend the expansion to terms of order  $m^5$ , we integrate by parts three times and then apply this Kummer transformation technique twice. We obtain

$$\begin{aligned}
\epsilon_n &= \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{8}{315} \sum_{k=n+1}^{\infty} \frac{1}{k^3(k^2-1)^3} - \frac{16}{315} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^4} \\
&\quad + \frac{128}{3465} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^5} - \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} + \frac{1024}{3003} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^{14}}{k(k^2-t^2)^7} dt \\
&= \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{32}{315} \sum_{k=n+1}^{\infty} \frac{1}{k^3(k^2-1)^4} + \frac{128}{3465} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^5} \\
&\quad - \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} + \frac{1024}{3003} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^{14}}{k(k^2-t^2)^7} dt \\
&= \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\
&\quad - \frac{32}{3465} \sum_{k=n+1}^{\infty} \frac{41k^2+3}{k^5(k^2-1)^5} - \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} + \frac{1024}{3003} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^{14}}{k(k^2-t^2)^7} dt.
\end{aligned}$$

To bound  $\epsilon_n$  from *above*, we observe that

$$\begin{aligned}
\epsilon_n &= \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\
&\quad - \left\{ \frac{32}{3465} \sum_{k=n+1}^{\infty} \frac{41k^2+3}{k^5(k^2-1)^5} + \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} - \frac{1024}{3003} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^{14}}{k(k^2-t^2)^7} dt \right\}.
\end{aligned}$$

We will show that the term in curly brackets is *positive*. In fact,

$$\begin{aligned}
&\frac{32}{3465} \sum_{k=n+1}^{\infty} \frac{41k^2+3}{k^5(k^2-1)^5} + \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} - \frac{1024}{3003} \sum_{k=n+1}^{\infty} \int_0^1 \frac{t^{14}}{k(k^2-t^2)^7} dt \\
&> \frac{32}{3465} \sum_{k=n+1}^{\infty} \frac{41k^2}{k^5(k^2-1)^5} + \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} - \frac{1024}{3003} \sum_{k=n+1}^{\infty} \frac{1}{(k-1)^{15}} \int_0^1 t^{14} dt \\
&= \frac{1312}{3465} \sum_{k=n+1}^{\infty} \frac{k^2}{k^5(k^2-1)^5} + \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)^6} - \frac{1024}{3003} \sum_{k=n+1}^{\infty} \frac{1}{15(k-1)^{15}} \\
&> \left( \frac{1312}{3465} + \frac{256}{9009} \right) \sum_{k=n+1}^{\infty} \frac{1}{k^{13}} - \frac{1024}{45045} \sum_{k=n+1}^{\infty} \frac{1}{(k-1)^{15}} \\
&= \sum_{k=n+1}^{\infty} \left[ \frac{6112}{15015} \frac{1}{k^{13}} - \frac{1024}{45045} \frac{1}{(k-1)^{15}} \right],
\end{aligned}$$

and a simple exercise in inequalities shows that each summand of the rightmost sum is positive whenever  $k > n \geq 3$ . Therefore,

$$\epsilon_n < \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5}.$$

To bound  $\epsilon_n$  from *below*, we note that

$$\begin{aligned} \epsilon_n &> \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\ &\quad - \left( \frac{32}{3465} \sum_{k=n+1}^{\infty} \frac{41k^2 + 3}{k^5(k^2 - 1)^5} + \frac{256}{9009} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2 - 1)^6} \right) \\ &> \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \left( \frac{32}{3465} \cdot 42 + \frac{256}{9009} \right) \sum_{k=n+1}^{\infty} \frac{1}{(k-1)^{13}} \\ &> \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \left\{ \frac{1}{2310m^5} - \frac{4688}{135135(n - \frac{1}{2})^{12}} \right\} \\ &> \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4}, \end{aligned}$$

since, as another simple exercise in inequalities shows, the term in curly brackets is positive for  $n \geq 5$ . Finally, the theorem can be checked directly for  $n = 1, 2, 3, 4$ .  $\square$

Now we note two fascinating corollaries due, in concept, but without the optimal error estimates, to the British mathematician ALFRED LODGE [5].

**Corollary 1.** *For every positive integer  $n$ , there is a  $\lambda_n$  for which*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m + \frac{6}{5}} + \lambda_n$$

where

$$0 < \lambda_n < \frac{19}{25200m^3}.$$

In fact,

$$\lambda_n = \frac{19}{25200m^3} - \rho_n,$$

where  $0 < \rho_n < \frac{43}{84000m^4}$ . The constants  $\frac{19}{25200}$  and  $\frac{43}{84000}$  are the best possible.  $\square$

**Corollary 2.** *For every positive integer  $n$ , define the quantity  $\Lambda_n$  by the following equation:*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} =: \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m + \Lambda_n}.$$

Then

$$\Lambda_n = \frac{6}{5} - \frac{19}{175m} + \frac{13}{250m^2} - \frac{\delta_n}{m^3},$$

where  $0 < \delta_n < \frac{187969}{4042500}$ . The constants in the expansion of  $\Lambda_n$  all are the best possible.  $\square$

**Corollary 3.** *For every positive integer  $n \geq 1$  there exists a number  $c_n$ ,  $0 < c_n < 1$ , such that the following approximation is valid:*

$$H_n = \frac{1}{2} \ln(2m) + \gamma + \frac{c_n}{12m}. \quad \square$$

The first and second corollaries appeared, in much less precise form and *with no error estimates*, in a very interesting paper by Lodge [5], which later mathematicians inexplicably ignored. Lodge gives some numerical examples of the error in the approximative equation

$$H_n \approx \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m + \frac{6}{5}}$$

in Corollary 1; he also presents the first two terms of  $\Lambda_n$  from Corollary 2. An asymptotic error estimate for Corollary 1 (with the incorrect constant  $\frac{1}{150}$  instead of  $\frac{1}{165\frac{15}{19}}$ ) appears as Exercise 19 on page 460 in Bromwich [2].

Our third corollary is the exercise (no. 18, page 460) that Bromwich originally proposed and is, of course, a trivial consequence of our main theorem. In fact, it is due to E. CESÀRO [3] who proved it in 1885 by a completely different technique. By the way, this was two years before Ramanujan was born!

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## References

- [1] B. Berndt, *Ramanujan's Notebooks, Volume 5*, Springer, New York, 1998.
- [2] T. J. l'A. Bromwich, *An Introduction to the Theory of Infinite Series*, Chelsea, New York, 1991.
- [3] E. Cesàro, “Sur la serie harmonique”, *Nouv. Ann. (3)* **IV** (1885), 295–296.
- [4] K. Knopp, *Theory and Application of Infinite Series*, Dover, New York, 1989.
- [5] A. Lodge, “An approximate expression for the value of  $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}$ ,” *Messenger of Mathematics* **30** (1904), 103–107.